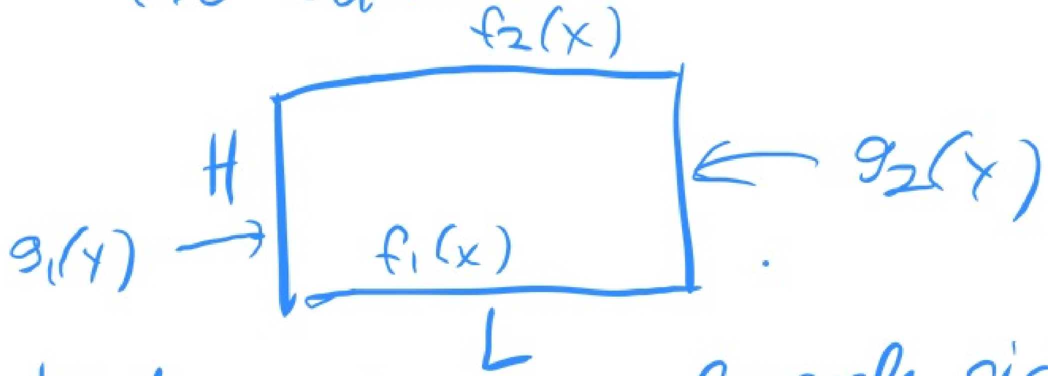
Laplace Equation for Rectangle

$$\nabla^2 u = 0 \iff \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

here $u = u(x,y)$

($\approx$ equilibrium cond. of heat equation in 2 dimensions)



four boundary cond., one for each side:

$$u(0,y) = g_1(y) \qquad u(x,0) = f_1(x)$$

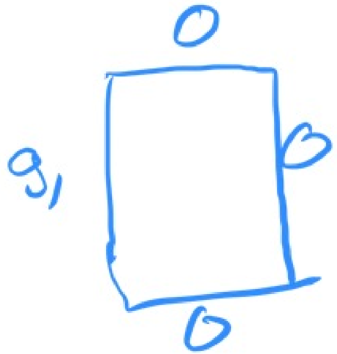
$$u(L,y) = g_2(y) \qquad u(x,H) = f_2(x)$$

nonhomogeneous boundary conditions!

Trick: we reduce this problem to four easier problems, where we make 3 of the four boundary conditions homogeneous.

example: $\nabla^2 u_4 = 0$

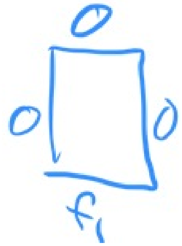
BC
$$u_4(0,y) = g_1(y)$$
$$u_4(L,y) = 0 \qquad \forall y$$
$$u_4(x,0) = 0 = u_4(x,H) \qquad \forall x$$



other cases:

e.g. u_1:

u_2:



etc.

Remark: assume we can solve these 4 problems, i.e. for u_1, u_2, u_3 and u_4

then the function $u = u_1 + u_2 + u_3 + u_4$

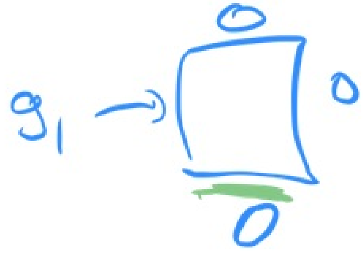
does solve our initial problem

(each u_i satisfies PDE

adding them up solves BC)

we consider the problem u_4:

$$u_4(0,y) = g_1(y)$$



separation of variables:

$$u_4(x,y) = h(x)\,\phi(y)$$

boundary conditions: $u_4(L,y) = 0 \quad \forall y$

$$\Updownarrow$$

$$h(L)\,\phi(y) = 0 \quad \forall y \quad \Longrightarrow \quad \boxed{h(L) = 0}$$

similarly: $u_4(x,0) = 0 \quad \forall x$

$$\overset{\parallel}{h(x)}\,\phi(0) = 0 \quad \forall x \quad \Longrightarrow \quad \boxed{\phi(0) = 0}$$

same way: $\boxed{\phi(H) = 0}$

plug $u_4(x,y) = h(x)\,\phi(y)$ into PDE

$$\Longrightarrow \frac{\partial^2 (h(x)\phi(y))}{\partial x^2} + \frac{\partial^2 (h(x)\phi(y))}{\partial y^2} = 0 \quad \Longrightarrow \quad \boxed{h''(x)\,\phi(y) + h(x)\,\phi''(y) = 0}$$

divide by $h(x)\,\phi(y)$

$$\frac{h''(x)}{h(x)} + \frac{\phi''(y)}{\phi(y)} = 0$$

$$\boxed{\frac{h''(x)}{h(x)} = -\frac{\phi''(y)}{\phi(y)} = \lambda}$$

wrong sign → only get 0 solution

why λ and not $-\lambda$

(• do try and error to find out which sign works)

or • our boundary cond. $\phi(0) = 0 = \phi(H)$ suggest we would need to get periodic solutions)

$$\Rightarrow \phi''(y) = -\lambda\phi(y)$$

$$\Rightarrow \boxed{\phi(y) = c_1 \cos\sqrt{\lambda}\, y + c_2 \sin\sqrt{\lambda}\, y}$$

boundary cond. $\phi(0) = 0 = \phi(H)$

$0 = \phi(H) = \sin\sqrt{\lambda}H \Rightarrow \sqrt{\lambda}H = n\pi$

(as in previous examples) $\Rightarrow \boxed{\lambda = \frac{n^2\pi^2}{H^2}}$

$\Rightarrow$ get $h''(x) = \frac{n^2\pi^2}{H^2} h(x)$

$$\Rightarrow \boxed{h(x) = c_1 e^{\frac{n\pi x}{H}} + c_2 e^{-\frac{n\pi x}{H}}}$$

BC: $0 = h(L) = c_1 e^{n\pi L/H} + c_2 e^{-n\pi L/H}$

$\Rightarrow c_2 = -e^{2n\pi L/H} c_1$

$\Rightarrow h(x) = c_1 \left(e^{\frac{n\pi x}{H}} - e^{2n\pi L/H} e^{-\frac{n\pi x}{H}} \right)$

Cosmetic change

$$h(x) = \underbrace{2c_1 e^{n\pi L/H}}_{=A} \cdot \underbrace{\frac{1}{2}\left(e^{\frac{n\pi}{H}(x-L)} - e^{-\frac{n\pi}{H}(x-L)}\right)}_{\sinh \frac{n\pi}{H}(x-L)}$$

$$\boxed{h(x) = A \sinh \frac{n\pi}{H}(x-L)}$$

result: get product solution

$$\boxed{u(x,y) = A \sinh \frac{n\pi}{H}(x-L) \sin \frac{n\pi}{H} y}$$

need to satisfy cond. $u_4(0,y) = g_1(y)$

try to find scalars A_n such that

$$\boxed{u_4(x,y) = \sum A_n \sinh \frac{n\pi}{H}(x-L) \sin \frac{n\pi}{H} y}$$

satisfies nontrivial boundary cond.

$$g_1(y) = u_4(0,y) = \sum A_n \sinh \frac{n\pi}{H}(-L) \sin \frac{n\pi y}{H}$$

use orthogonality of sine functions

$$\underbrace{\left(g_1(y), \sin \frac{k\pi y}{H}\right)}_{= \int_0^H g_1(y) \sin \frac{k\pi y}{H} dy} = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{H}(-L) \underbrace{\left(\sin \frac{n\pi y}{H}, \sin \frac{k\pi y}{H}\right)}_{=0 \text{ for } k \neq n}$$

$$= A_k \sinh \frac{k\pi}{H}(-L) \cdot \underbrace{\left(\sin \frac{k\pi y}{H}, \sin \frac{k\pi y}{H}\right)}_{= H/2}$$

previous calculation

$$\boxed{\text{here: } (f,g) = \int_0^H f(y) g(y)\, dy}$$

solve for A_k:

$$\boxed{A_k = \frac{2}{H \sinh \frac{k\pi}{H}(-L)} \int_0^H g_1(y) \sin \frac{k\pi y}{H} dy}$$

$\Rightarrow$ get solution $u_4(x,y)$

$\Rightarrow$ applying same strategy for u_1, u_2, u_3

get solution $u(x,y)$ of original problem.